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SHALLOW-WATER EQUATIONS WITH DISPERSION. HYPERBOLIC MODEL

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A hyperbolic shallow-water model is constructed with allowance for nonlinear and dispersive effects. The model describes solitonlike solutions in a range of wave velocities and predicts the breaking of smooth waves when the limiting amplitude is attained. The model is found to be adequate by comparison with experimental data on the evolution of a wave packet generated by the moving lateral wall of a channel.

Introduction. The presence of solitary waves is the characteristic property of a nonlinear dispersive dissipation-free medium. In such media, the tendency to wave breaking, which is caused by the nonlinear character of the equation, is compensated by dispersion, thus providing for the existence of invariable-shape waves. Another distinguishing property, namely, the additional law of energy conservation, leads to the absence of structures, i.e., running waves with different limits at infinity, among the solutions. In some cases, media with dispersion are described by continuum equations in which the equation of state relates the pressure not only to thermodynamic variables but also to their derivatives. Such equations appear, for example, in the theory of long waves on the surface of a liquid [1] and in the models of two-phase media [2]. The variational principle of derivation of these equations was discussed by Gavrilyuk and Shurgin [3].

If the equation of state includes derivatives with respect to the desired functions of spatial variables, the system of equations that describes the motion is not hyperbolic. Various versions of the Boussinesq and Korteweg–de Vries equations which appear in the second shallow-water approximation [1] fall into this class of equations.

However, the motions of a dispersive medium can be given by a hyperbolic system in the case where only the derivatives with respect to the internal parameters of the medium, calculated along the particle trajectory, enter the equation. For example, for a single-velocity model of a bubbly liquid, the system of Iordanskii–Kogarko equations [4] for a compressible carrying medium is hyperbolic. The more general hyperbolic model of a two-phase medium was developed by the author in [5].

The main difference of the first-order hyperbolic systems that describe the media with dispersion from the higher-order systems is the fact that smooth solitonlike solutions in the first case exist only in some range. In propagating a nonlinear wave with a velocity exceeding the velocity of long waves, a structure of the jump–wave type is formed, i.e., after the limiting amplitude is attained, the solitons break in the hyperbolic models.

With the limiting amplitude attained, the problem of wave breaking is of particular interest in the theory of gravity waves. Although a correct theory allows one to obtain the limiting amplitude and velocity of wave propagation in the Cauchy–Poisson problem, the soliton amplitude can be arbitrary [1, 6] in the second approximation of shallow-water equations. The upper boundaries of the soliton velocity, which arise in some models, are not internal for this model, but they are introduced from the exact formulation of the problem. For example, Ovsyannikov et al. [6] showed that the velocity of the solitons which propagate in a liquid at rest with depth h and acceleration of gravity g is not higher than $c = \sqrt{2gh}$ within the framework of the second

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approximation. This estimate was derived from the wave's shape determined from the second approximation and the Bernoulli integral taken from the correct formulation of the problem.

The goal of the present paper is to construct the simplest hyperbolic shallow-water model with dispersion in which there are not only the conventional critical velocity of long waves $c_1 = \sqrt{gh}$, but also the second critical velocity $c_2 = \beta\sqrt{gh}$ ($\beta > 1$), which corresponds to the propagation velocity of the waves of limiting amplitude. The velocity c_2 was determined experimentally by Bukreev and others [7, 8], who showed that a transition from smooth to broken waves as the critical velocity c_2 is reached is observed both for the solitary waves and for the bores which propagate in an unperturbed liquid.

Since the model that we constructed contains no empirical constants, except for the factor α, which characterizes the ratio of the vertical and horizontal scales of motion, the adequacy of the model was established by comparison with concrete experimental data on the evolution of a wave packet generated by a moving lateral wall [8].

1. Mathematical Model. Shallow-water equations are derived for the mean depth $h(t,x)$ and the velocity $u(t,x)$ under the assumption of the hydrostatic character of the pressure distribution in a liquid layer. Taking account of the nonhydrostatic character can be performed by introducing a new unknown variable η that characterizes the instantaneous value of the layer depth, which can be different from the average value of h. Therefore, the total pressure P in the equations of motion is regarded as a function of variables h and η. To determine the new variable η, we use an additional energy equation in which the internal energy ε depends on the variables h and η and the velocity

$$v = \eta_t + u\eta_x. \tag{1.1}$$

Note that $\varepsilon_h(h,\eta,v) = P/h^2$.

For the one-dimensional motion of a homogeneous incompressible-fluid layer in the gravity field, the laws of conservation of mass, momentum, and energy are of the form

$$\begin{gathered} h_t + (hu)_x = 0, \qquad (hu)_t + (hu^2 + P)_x = 0, \\ (h(u^2/2 + \varepsilon))_t + (hu(u^2/2+\varepsilon) + Pu)_x = 0. \end{gathered} \tag{1.2}$$

After that, we consider flows in which the pressure distribution in a fluid layer differs little from the hydrostatic distribution, i.e., $|\eta - h|/h \ll 1$. Therefore, the concrete form of the dependence $P(\eta, h)$ is of no concern. It is important to specify correctly the behavior of this function in the neighborhood of the equilibrium state $\eta = h$.

We shall choose the following dependences:

$$P = \frac{1}{2} gh^2 \left(\frac{h}{\eta}\right), \quad \varepsilon = \frac{1}{4} g \frac{h^2}{\eta} + \frac{1}{4} g\eta + \alpha v^2 \tag{1.3}$$

which ensure the hydrostatic pressure distribution in an equilibrium flow for $h \equiv \eta$.

Of course, the choice of the equations of state (1.3) is not unambiguous, but the form of the function $\varepsilon(h,\eta,v)$ is found with accuracy of up to the constant factor α from the dimension of the entering quantities for a given dependence $P(h,\eta)$. The value of α is not important either, because the parameter α can be excluded from system (1.1)–(1.3) by using the extension of the variables which is used in modeling long-wave flows: $x \to x$, $t \to \varepsilon^{-1/2} t$, $h \to \varepsilon h$, and $\eta \to \varepsilon\eta$.

The consequence of system (1.1)–(1.3) is an analog of the Rayleigh equations for bubble oscillations in a fluid [9]:

$$v_t + uv_x = (g/8\alpha)(h^2/\eta^2 - 1).$$

In view of this, the usual shallow-water equations, i.e., the first two equations in (1.2) with $P = (1/2)gh^2$ and the equilibrium characteristics $\lambda_e^{\pm} = u \pm \sqrt{gh}$, are the equilibrium model ($h \equiv \eta$ and $v \equiv 0$) for the derived system. The characteristics of system (1.1)–(1.3) in an equilibrium flow ($h = \eta$) can be represented in the form $\lambda_f^{\pm} = u \pm \sqrt{1.5gh}$. In addition, there is the multiple contact characteristic $\lambda_f^0 = u$.

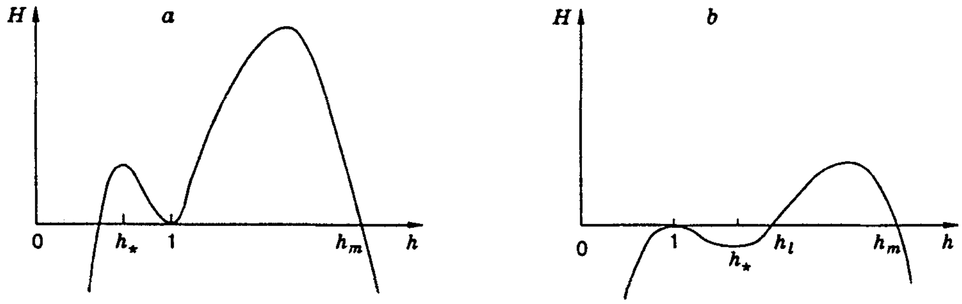

Fig. 1

Thus, we are dealing with the classical situation where the characteristics of the equilibrium and original models alternate, i.e., $\lambda_f^- < \lambda_e^- < \lambda_f^0 < \lambda_e^+ < \lambda_f^+$. Therefore, smooth running waves for a complete system exist within either $\lambda_e^+ < D < \lambda_f^+$ or $\lambda_f^- < D < \lambda_e^-$ (see [1]).

2. Running Waves of System (1.1)–(1.3). We shall consider the solutions of (1.1)–(1.3) that depend on the variable $\xi = x - Dt$ ($D > 0$). We note that, for $\xi \to \infty$, the solution tends to an equilibrium state ($h = \eta = h_0$, $v = 0$, and $u = 0$) The single dimensionless parameter that determines the wave structure is the Froude number $\mathrm{Fr} = D/\sqrt{gh_0}$. Passing over to dimensionless variables, one can assume that $h_0 = 1$, $g = 1$, and $\mathrm{Fr} = D$.

The conservation laws (1.2) yield the following wave relations:

$$h(u-D) = -D, \quad h(u-D)^2 + \frac{1}{2}\frac{h^3}{\eta} = D^2 + \frac{1}{2},$$
$$\frac{1}{2}(u-D)^2 + \frac{3}{4}\frac{h^2}{\eta} + \frac{1}{4}\eta + \alpha v^2 = \frac{1}{2}D^2 + 1. \tag{2.1}$$

The dependences $\eta = \eta(h)$ and $v^2 = (1/\alpha)H(h)$ are found from (2.1), and the wave shape is restored from Eq. (1.1), which takes the following form for a running wave:

$$\frac{d\eta(h)}{d\xi} = -\frac{hv(h)}{D}. \tag{2.2}$$

The functions $\eta(h)$ and $H(h)$ are defined explicitly from (2.1):

$$\eta(h) = h^4/(2hD^2 + h - 2D^2), \quad H(h) = 1 - \frac{3}{4}\frac{h^2}{\eta(h)} - \frac{1}{4}\eta(h) - \frac{1}{2}D(h^{-2} - 1). \tag{2.3}$$

The function $a(h) = d\eta/dh$ has the single root $h_* = 8D^2/3(1+2D^2)$, which corresponds to a minimum of the function $\eta(h)$. We note that $h_* > 1$ only if $D > \sqrt{1.5} = \lambda_f^+$. The necessary condition for the existence of the continuous solution (2.1) and (2.2) is the positiveness of the function $H(h)$ from (2.3) in the neighborhood of $h = 1$. By virtue of (2.1), we have

$$\frac{dH}{dh} = \frac{1}{4}\Big(\frac{h^2}{\eta(h)^2} - 1\Big)a(h),$$

and the behavior of the function $H(h)$ in the neighborhood of $h = 1$ is determined by the value of the derivative $(d\eta/dh)(1) = a(1) = 3 - 2D^2$. The inequalities $0 < a(1) < 1$, which ensure the positiveness of the function $H(h)$ in the vicinity of $h = 1$, are satisfied if $1 < D < \sqrt{1.5}$. For this case, the graph of the function $H = H(h)$ is depicted in Fig. 1a. For $h > 1$, the solution of system (2.1) and (2.2) is a soliton whose velocity is in the interval between the equilibrium and frozen velocities of the characteristics. The positive part of the function $H(h)$ for $h < 1$ produces no soliton, because the solution of Eq. (2.2) cannot be extended through the point $h_* < 1$ at which $a(h_*) = 0$.

A smooth soliton of the limiting amplitude is implemented for $D = \sqrt{1.5}$. Here we have $h_m = 1.45$ and $\eta_m = \eta(h_m) = 1.58$. For $D > \sqrt{1.5}$, only a configuration of the jump-wave type [9] is possible. This configuration consists of a hydraulic jump which transforms $h = 1$ into $h = h_1$ [we note that $H(h_1) > 0$, i.e., $h_l < h_1 < h_m$] and a periodic solution with minimum depth h_l and maximum depth h_m (see Fig. 1b). Since Eq. (1.1) is written in a nondivergent form, the question arises on the choice of relations on a hydraulic jump, which determine the state behind the jump.

In the present study, we shall not dwell upon this problem in detail. We note only that, for nonuniform systems of the type (1.1)–(1.3), the choice of the laws of conservation is of no primary concern because the variation of the value of h_1 behind the jump front leads only to the phase shear of the periodic solution following the wave, and the problem of the choice of relations on a discontinuity reduces to the question which section of the stationary periodic wave behind the jump should be included in a section of the relaxation zone that is replaced by the discontinuity in this model. Therefore, one can assume that the values of h_l or h_m are replaced behind the jump at which the velocity is $v = 0$. Here Eq. (1.1) is written in the form of an inhomogeneous conservation law:

$$\frac{\partial hv}{\partial t} + \frac{\partial huv}{\partial x} = F(h, \eta, v). \tag{2.4}$$

The function F is chosen in such a manner that Eq. (2.4) is a differential consequence of system (1.1)–(1.3).

3. Comparison with Experiment [8]. As mentioned in deriving system (1.1)–(1.3), the parameter α appears in the modeling and has the order ε^2. The numerical value of α is obtained from comparison with experiment. One effective and controllable method of obtaining a wave packet is wave generation of the moving lateral wall of a channel [7, 8]. This problem is completely equivalent, within the framework of the constructed model, to the problem of the motion of a piston in a gas at rest according to a prescribed law. In contrast to the gas dynamics, the wave structure, however, will be much more complicated because of the nonlinear dispersion. To test the model, we shall use the experimental data of [8] in which the velocity of the wall was maintained close to the constant velocity U for ΔT, and the wall immediately stopped. As a result, an analog of the N wave was formed at a fairly long distance from the moving wall, but the initially monotone perturbation disintegrated into a chain of solitons moving with different velocities because of dispersion effects. In [8], the attention was mainly focused on the experimental determination of the critical velocity c_2 at which the initially smooth waves break. The value of $c_2 = \lambda_f^+ = \sqrt{1.5gh_0}$ at which wave breaking occurs in the model (1.1)–(1.3) is in agreement with that obtained in [8]. The same can be shown for the limiting amplitude $\eta_m = 1.58\sqrt{gh_0}$, which was obtained in Sec. 2 for a smooth soliton propagating at the limiting velocity $\lambda_f^+ = \sqrt{1.5gh_0}$ over a quiescent liquid of depth h_0.

However, it is of keen interest to compare the phase pattern of waves generated by a moving wall with experimental data. The problem is essentially nonstationary, and its solution can be found only by a numerical calculation by the model (1.1)–(1.3).

Since the structure of the equations is similar to the equations of gas dynamics, one can use the standard calculation schemes. We use an analog of Godunov's scheme, and, to solve the inhomogeneous equations along the trajectories, use is made of the Runge–Kutta method with a time step matched with the basic scheme. Calculation results for $U = 23.3$ cm/sec, $\Delta T = 1.3$ sec, and $h_0 = 3.1$ cm are given in Fig. 2 in the form of the dependence $\eta = \eta(t)$ (solid curve) for fixed values of x, which are reckoned from the initial point of motion of the wall: $x = 50$, 110, and 250 cm (Fig. 2a–c). The dotted curve shows the experimental dependences from [8, Table 1] for the initial and boundary conditions indicated above. It is worth noting that the time of motion of the wall in a numerical calculation is decreased compared with the experimental value ($\Delta T \sim 1.76$ sec) to obtain the same duration of the initial perturbation immediately after stoppage of the wall ($x = 50$ cm). The quantity α that characterizes the ratio of the vertical to the horizontal scale equals 0.025.

The effect of the dissipation associated with wall friction, viscosity, and wave breaking is not taken into account in the model. Nevertheless, the numerical calculation incorporates qualitatively and quantitatively the major specific features of the evolution of the nonlinear wave packet.

Of interest is the process of development of a wave packet from a local initial perturbation which is

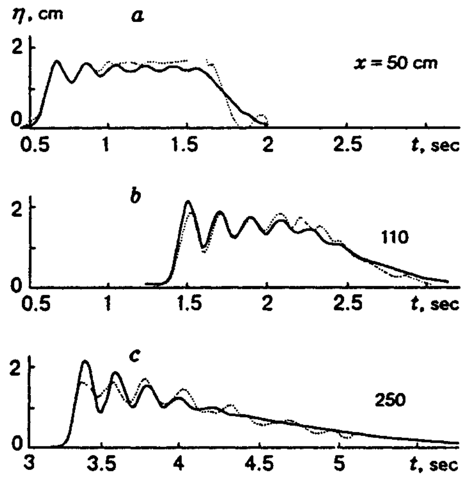

Fig. 2

close to a piecewise-constant perturbation. With distance from the source, the perturbation disintegrates into a chain of waves whose amplitude grows. The amplitude of the first crest attains the limiting one already at a distance of $x = 110$ cm, and one can observe the development of the wave-breaking process in the experiment which causes a decrease in the amplitude of the leading wave [8]. The mechanism of wave breaking is not envisaged in the model (1.1)–(1.3), and hence the numerical and experimental values of the amplitudes of the leading wave differ greatly at $x = 250$ cm. Nevertheless, the phase and amplitude characteristics of the remaining waves in the wave packet are reflected satisfactorily. This means that despite the presence of powerful dissipative mechanisms, the main contribution to the formation of the wave packet is given by linear and dissipation effects, which are correctly reflected in this model.

4. Conclusions. The shallow-water model constructed possesses a number of advantages. It is an inhomogeneous hyperbolic system of equations for which the first-order equations in the shallow-water theory are an equilibrium model. In view of this, the model contains solitonlike solutions propagating in a range of velocities ($\sqrt{gh_0} < D < \sqrt{1.5gh_0}$) over a quiescent liquid of depth h_0. If the soliton velocity exceeds the second critical velocity (see [8]), which coincides in the model considered with the velocity of the characteristics, i.e., $D > \sqrt{1.5gh_0}$, there is no smooth solution of system (1.1)–(1.3). This model does not describe the wave-breaking process upon attainment of the limiting amplitude. However, the combination of the approaches developed in the present paper with the possibility of considering the near-surface flow region as an interlayer in which an intense short-wave or turbulent motion with a set of parameters of its own that characterize it completely [10] indicates a method of constructing a more complete model. Both the dispersion effects and the effect of turbulent mixing on the structure of large-amplitude gravity waves should be equally presented in an extended model. This offers the possibility of an adequate description of the transition of a wave bore to a turbulent one as its temperature rises.

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REFERENCES

1. G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley and Sons, New York (1974).
2. V. E. Nakoryakov, "Hydrodynamics of two-phase flows," in: *Hydrodynamics and Heat Transfer in Two-Phase Media* (Collected scientific papers) [in Russian], Inst. of Thermal Physics, Sib. Div., Acad. of Sci. of the USSR, Novosibirsk (1981).
3. S. L. Gavrilyuk and S. M. Shurgin, "Media with equations of state that depend on derivatives," *Prikl. Mekh. Tekh. Fiz.*, **37**, No. 2, 35–49 (1996).

4. S. V. Iordanskii, "On the equations of motion of a liquid with gas bubbles," *Prikl. Mekh. Tekh. Fiz.*, No. 3, 102–110 (1960).
5. V. Yu. Liapidevskii, "Modeling of two-phase flows on the basis of conservation laws," in: *Dynamics of Continuous Media* (Collected scientific papers) [in Russian], Novosibirsk, **76** (1986), pp. 111–120.
6. L. V. Ovsyannikov, N. I. Makarenko, V. I. Nalimov, et al., *Nonlinear Problems of the Theory of Surface and Internal Waves* [in Russian], Nauka, Novosibirsk (1985).
7. V. I. Bukreev and N. P. Turanov, "Experiments with shallow-water waves generated by the motion of the lateral wall of a channel," *Prikl. Mekh. Tekh. Fiz.*, **37**, No. 6, 44–50 (1996).
8. V. I. Bukreev, E. M. Romanov, and N. P. Turanov, "Breaking of gravity waves in the neighborhood of the second critical velocity of their propagation," *Prikl. Mekh. Tekh. Fiz.*, **39**, No. 2, 52–58 (1998).
9. V. Yu. Liapidevskii and S. I. Plaksin, "Structure of shock waves in a gas–liquid medium with the nonlinear equation of state," in: *Dynamics of Continuous Media* (Collected scientific papers) [in Russian], Novosibirsk, **62** (1983), pp. 75–92.
10. V. Yu. Liapidevskii, "Flow chocking in flowing about an obstacle by a two-layer mixed liquid," *Prikl. Mat. Mekh.*, **58**, No. 4, 108–112 (1994).